On the Bayesian Nonparametric Estimation of Discontinuous Densities

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October 31, 2018

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Introduction

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Estimation of a density f_0 supported on a compact metric space $(\mathrm{M}; d)$ using observations $f X_i g_{i=1}^k$ i.i.d. f_0.

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The focus for this talk:

What happens with Bayesian nonparametric procedures when f_0 is discontinuous?

Motivation

Density f_0 supported on a compact metric space $\mathbb{M}$:

Analysis of non-euclidean data; Statistics on manifolds.

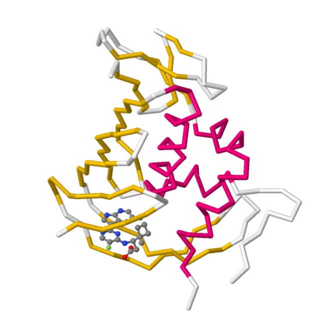
$\mathbb{M}$ could be the circle, the sphere, a quotient space, etc.

We want to account for the particular metric structure of $\mathbb{M}$.

Motivation

Bioinformatics:

Protein structure prediction.



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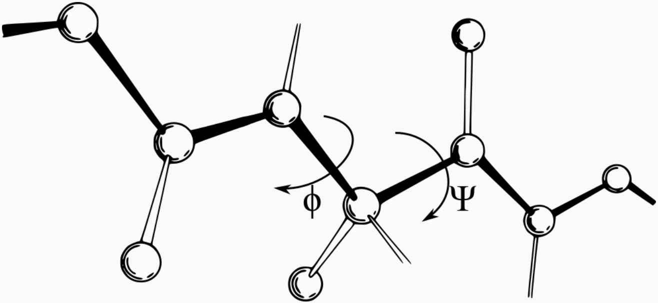

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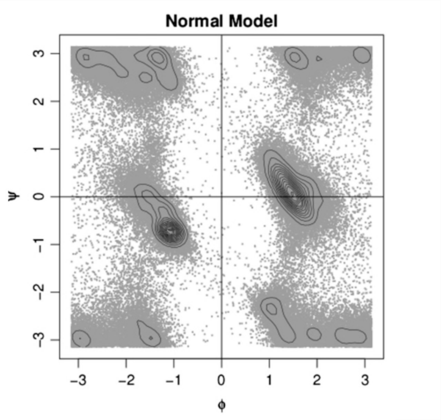
Estimation of angle pairs distributions using mixtures of normal distributions (Dhal & al. (2008) *J. Mol. Bio.*).



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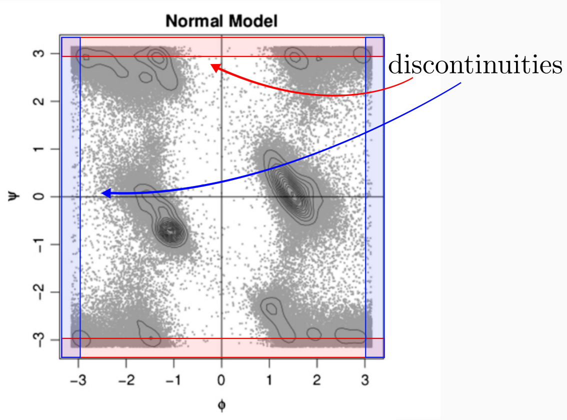

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The team adjusted their model the next year (Lennox & al. (2009) *JASA*).

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Case $M = [0; 1]$: Petrone and Wasserman (2002) show posterior consistency for random Bernstein polynomial priors at continuous densities.

Wu and Ghosal (2008) study the Kullback-Leibler property of sieve priors, but also require continuity.

Constrasts with KDE which are known to be L^1 consistent at bounded discontinuous densities since Devroye & Wagner (1979).

Motivation

What happens when we drop continuity assumptions on f_0?

Contribution

Our contribution

Show that posterior consistency also holds at discontinuous (bounded) densities for a class of sieve priors.

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Our contribution

Show that posterior consistency also holds at discontinuous (bounded) densities for a class of sieve priors.

In particular, we show that continuity assumptions are not necessary for the random (Bernstein) polynomial priors of Petrone and Wasserman (2002).

Background

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F as set of densities; a prior on F.

Kullback-Leibler divergence between f_0 and f:

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$\mathcal{F}$ as set of densities; Π a prior on $\mathcal{F}$.

Kullback-Leibler divergence between f_0 and f:

$$K(f_0; f) = \int_{\{f_0 > 0\}} f_0 \log \frac{f_0}{f}.$$

Why is it important? If $\{X_i\}$ is i.i.d. f_0, then

$$K(f_0; f) = L \iff \prod_{i=1}^{k} \frac{f_0(X_i)}{f(X_i)} = e^{Lk + o(k)} \quad \text{a.s. as } k \to \infty.$$

The Kullback-Leibler support of Π is the set of all densities $f_0 \in \mathcal{F}$ such that

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The posterior distribution of Π is strongly consistent at every density of its Kullback-Leibler support provided a complexity constraint on the prior (fundamental work by Andrew Barron and many others; see also Xing & Ranneby (2009)).

Strong posterior consistency means that if $\{X_i\}$ is i.i.d. f_0, then

$$\Pi\left(f : \int |f_0 - f| < \varepsilon \;\middle|\; \{X_i\}_{i=1}^{k} \right) \to 1$$

a.s. as $k \to \infty$ and for every $\varepsilon > 0$.

Implies almost sure L^1 convergence of the Bayes estimator.

The Kullback-Leibler support of sieve priors

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- $\mathcal{F}$ a set of bounded densities.
- Linear operators $T_n : L^1(M) \to L^1(M)$, $n \in \mathbb{N}$ which maps densities to densities and $T_n(\mathcal{F}) \subset \mathcal{F}$.

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We thus obtain a model $\mathcal{C} = \bigcup_{n \in \mathbb{N}} \mathcal{C}_n$ with $\mathcal{C}_n := T_n(\mathcal{F})$.

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- Linear operators $T_n : L^1(M) \to L^1(M)$, $n \in \mathbb{N}$ which maps densities to densities and $T_n(F) \subset F$.

We thus obtain a model $C = \bigcup_{n \in \mathbb{N}} C_n$ with $C_n := T_n(F)$. Priors Π_n on C_n and distribution ρ on $\mathbb{N}$ yields

$$\Pi = \sum_n \rho(n) \Pi_n.$$

- T_n is linear, maps densities to densities and $T_n(F) \subset F$.
- $= \sum_n (n) \;_n$.

Theorem (Binette & Guilotte, 2018)

Let F, $_n$, and T_n be as before. Suppose also that

- $T_n(F)$ is of finite dimension, and
- $\| T_n f - f \|_\infty \to 0$ for every continuous f.

If $(n) > 0$ and $_n$ has (L^1) support $T_n(F)$, then any (bounded) $f_0 \in F$ is in the Kullback-Leibler support of .

Corollary

Under the hypotheses of the Theorem, suppose further that $(n) < Ce^{cd_n}$ for some increasing sequence $d_n > \dim T_n(F)$ and some $c; C > 0$. Then the posterior distribution of is strongly consistent at any (bounded) f_0 2 F.

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Idea: If $(n) < Ce^{cd_n}$, then the prior satis es a nite Hausdor -entropy conditions which entails posterior consistency on the Kullback-Leibler support (Xing & Ranneby (2009); same spirit as Walker (2004) and Barron (1986)).

What can this be applied to?

- Sieve priors.
- Random series priors / nite mixtures with a random number of components.
- Dirichlet Process location mixtures with a discrete scale parameter; $T_n f = f \quad {}_{1=n}$.

Random polynomial priors

The Bernstein-Kantorovich operator is defined as

$$T_n f(x) = (n+1) \sum_{i=0}^{n} \int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} f(t)\,dt\, p_{i;n}(x); \quad x \in [0;1];$$

where $p_{i;n}(u) = \binom{n}{i} u^i (1-u)^{n-i}$ is the ith Bernstein polynomial of degree n.

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where $p_{i;n}(u) = \binom{n}{i} u^i (1-u)^{n-i}$ is the ith Bernstein polynomial of degree n. It may be extended to act on a probability measure P by letting

$$T_n P = (n+1) \sum_{i=0}^{n} P\left(\frac{i}{n+1}; \frac{i+1}{n+1}\right) p_{i;n}(x):$$

If P has a density f, then $T_n P = T_n f$.

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Bernstein-Dirichlet prior of Petrone and Wasserman (2002): DPM induced by the random density

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T_n is a linear operator mapping densities to densities; its image is of dimension n and it is well-known that $\|T_n f \quad f\|_\infty \to 0$ $\forall f \in C([0;1])$.

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Provided $T_n(D)$ has full support and $0 < \; (n) < Ce^{\; cn}$, then posterior consistency holds at any bounded f_0.

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Bernstein-Dirichlet prior of Petrone and Wasserman (2002): DPM induced by the random density

$$T_N D; \quad \text{where } N \sim \lambda; \; D \sim \text{DP}.$$

T_n is a linear operator mapping densities to densities; its image is of dimension n and it is well-known that $\|T_n f - f\|_\infty \to 0$ $\forall f \in C([0,1])$.

Provided $T_n(D)$ has full support and $0 < \lambda(n) < Ce^{-cn}$, then posterior consistency holds at any bounded f_0.

This shows the continuity assumptions of Petrone and Wasserman (2002) are not necessary; boundedness is sufficient.

Conclusion

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For a class of sieve priors, L^1 posterior consistency also holds at bounded discontinuous densities.

- Circular/spherical density estimation
- Copula density estimation and other shape constrained problems.

Notes and slides can be found on my personal page `olivierbinette.ca`.

Full story is in *Bayesian Nonparametrics for Directional Statistics* (Binette & Guillotte, 2018; arXiv:1807.00305).

Thank you!